Dynamic localization of Lyapunov vectors in Hamiltonian lattices

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The convergence of the Lyapunov vector toward its asymptotic shape is investigated in two different one-dimensional Hamiltonian lattices: the so-called Fermi-Pasta-Ulam and Φ^4 chains. In both cases, we find an anomalous behavior, i.e., a clear difference from the previously conjectured analogy with the Kardar-Parisi-Zhang equation. The origin of the discrepancy is eventually traced back to the existence of nontrivial long-range correlations both in space and time. As a consequence of this anomaly, we find that, in a Hamiltonian lattice, the largest Lyapunov exponent is affected by stronger finite-size corrections than standard space-time chaos.

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I. INTRODUCTION

Lyapunov exponents are the most direct indicators and quantifiers of deterministic chaos. From a knowledge of them, one can extract information about the invariant measure and the production of dynamical entropy. In spatially extended systems, the problem of characterizing the evolution of infinitesimal perturbations is made more complex (but also more interesting) by the need of looking at propagation and diffusion properties. This has given rise to a series of approaches where either the growth rate in moving frames has been introduced (velocity-dependent exponents [1]), or the propagation of perturbations with exponential spatial profiles has been discussed [2]. However, even without pretending to introduce new indicators or without going to higher-order Lyapunov exponents, the largest one already contains some interesting statistical information. In Ref. [3], we indeed showed that the spatial shape of the maximally expanding perturbation (hereafter called the Lyapunov vector for simplicity) follows the same dynamical laws as Kardar-Parisi-Zhang (KPZ) type interfaces [4]. Such a general analogy has several interesting consequences: we could quantify the amount of finite-size corrections (both in time and space), finding, for instance, that deviations from the thermodynamic limit are of the order $1/L$, where L is the lattice length. Another interesting consequence of the analogy is the observation that the logarithmic norm is a more appropriate way of estimating the largest Lyapunov exponent, as it averages the contributions coming from all spatial regions including those where the Lyapunov vector is temporarily very small. Contrary to this, when calculating the largest Lyapunov exponent with the usual L_2 norm, the main weight comes from the region where the Lyapunov vector appears to be temporarily localized.

The only one-dimensional models where the full correspondence with the scaling behavior of the KPZ equation does not hold are Hamiltonian chains. Understanding the origin of such a difference is precisely the task of this paper. Our previous analysis of coupled standard maps [3], revealing a nice correspondence with KPZ dynamics, already suggests that the difference should not be traced back to the symplectic structure (in contrast to the dissipative nature of all other models that we have studied). In fact, we shall see that the main reason for the different scaling exponents is due to the ubiquitous existence of long-range space-time correlations in one-dimensional Hamiltonian systems. In order to show this, we perform thorough numerical studies of two Hamiltonian models—Fermi-Pasta-Ulam (FPU) and Φ^4 lattices. Both are autonomous Hamiltonian systems, demonstrating space-time chaos (at least for the considered values of the energy density). In our study, we start with the statistical properties of the Lyapunov vectors, and demonstrate the deviations from the Kardar-Parisi-Zhang universality class. Then we investigate the origin of these deviations. In principle, the ''special'' symplectic structure of the equations could be responsible for this anomaly. However, we exclude this possibility by showing that the scaling predicted by the KPZ equation is recovered as soon as the fluctuating term in the tangentspace equations is replaced by a short-correlated stochastic process. Thus the anomalous scaling of Lyapunov vectors in Hamiltonian lattices is due to the long-range correlations of some observables, and the existence of such observables in typical chaotic Hamiltonian chains is the main result of our work.

A further interesting consequence of the anomalous scaling concerns the finite-size corrections affecting the largest Lyapunov vector. In fact, the deviations from the thermodynamic limit scale approximately as $1/\sqrt{L}$, and are thus much larger than in standard space-time chaos, where they are on the order of $1/L$. Such a difficulty must be borne in mind, especially when an accurate comparison with analytical predictions is attempted [5]. It is precisely this slow convergence that, in the absence of compelling theoretical arguments, has been misjudged as evidence of a logarithmic divergence in Ref. [6].

In Sec. II we briefly summarize the formalism needed to interpret the Lyapunov vector as a ''roughening interface,'' and recall the key consequences of the analogy. In Sec. III, we introduce the two models. Section IV is devoted to a thorough discussion of the numerical results in both cases, including finite-size corrections. In Sec. V we search the origin of the anomalies found and discussed in the previous one. Section VI deals with a direct investigation of space-time correlation properties of the proper observables. Finally, in Sec. VII, we summarize the main open problems.

II. LYAPUNOV VECTORS AS INTERFACES

In this section we briefly recall the results derived in Ref. [3] for the scaling behavior of the Lyapunov vector in space-time chaos, and introduce the necessary formalism to discuss the problem in the next sections. We consider a field $\mathbf{u}(x,t)$ that depends on one spatial coordinate $x\in[0,L]$ and time t, and that obeys some nonlinear evolution equations yielding space-time chaos. For calculating the maximal Lyapunov exponent, one has to follow the evolution of an infinitesimal perturbation $\mathbf{w}(x,t)$, obeying the linearized equations around the chaotic solution $\mathbf{u}(x,t)$. It is convenient to introduce the logarithmic amplitude of an infinitesimal perturbation as a scalar field

$$H(x,t)=\log||\mathbf{w}(x,t)||, \tag{1}$$

where $||\cdot||$ is some spatially local norm. In Ref. [3] we have found that in a rather broad class of systems, $H(x,t)$ exhibits the scaling behavior predicted for roughening interfaces. More precisely, $H(x,t)$ belongs to the universality class of the Kardar-Parisi-Zhang equation

$$h_t=Dh_{xx}+\nu(h_x)^2+v+\xi(x,t), \tag{2}$$

where $\xi(x,t)$ is a zero-average δ-correlated (both in space and time) process, and D, ν, and v are suitable effective parameters. Although we cannot explicitly derive an equation of like Eq. (2) for the field $H(x,t)$, in Ref. [3] we demonstrated that the scaling properties of $H(x,t)$ for large system sizes L and large observation times t coincide with those of the solution of Eq. (2). Below we recall these properties.

It is known that the width

$$W(L,t)=\sqrt{\langle h^2\rangle-\langle h\rangle^2} \tag{3}$$

of an initially flat interface evolves, for sufficiently large L and t, as

$$W(L,t)\approx L^{\alpha}F\left(\frac{t^{\beta}}{L^{\alpha}}\right), \tag{4}$$

with $\alpha=1/2$ and $\beta=1/3$. The scaling function $F(y)$ behaves linearly for small y, and converges to a constant value for $y\to\infty$. This implies that the interface profile converges, asymptotically in time, to a Brownian motion, since $W(L,\infty)\approx\sqrt{L}$.

Now we want to derive the expression for the scaling behavior of finite-size and finite-time corrections to the Lyapunov exponent. The Lyapunov exponent λ is nothing but the average velocity $\langle H_t\rangle$. From Eq. (2), it is clear that the dependence of λ on L and t is entirely contained in the nonlinear term

$$\lambda(L,t)=\langle h_t\rangle=v+\nu\langle(h_x)^2\rangle. \tag{5}$$

In Fourier space, $\Delta\lambda:=\lambda(L,t)-v$ can be expressed as

$$\Delta\lambda=2\nu\int_{1/L}^{k_{max}}k^2S(k,t)dk, \tag{6}$$

where $S(k,t)=\langle|h_k|^2\rangle$ is the spatial power spectrum of $h(x,t)$ with an ultraviolet cutoff at k_{max} (given, in the present case, by the inverse of the lattice spacing).

For an initially flat profile, the power spectrum $S(k,t)$ evolves in time according to

$$S(k,t)\sim k^{\gamma}\theta(kt^{\delta}-\mathrm{const}). \tag{7}$$

Calculation of the interface width according to

$$W^2(L,t)=\int_{1/L}^{\infty}S(k,t)dk,$$

and comparison with Eq. (4), gives

$$\gamma=-1-2\alpha,\quad \delta=\beta/\alpha. \tag{8}$$

By substituting Eq. (7) into Eq. (6), we obtain

$$\Delta\lambda(L,t)\approx L^{2\alpha-2}\left[\tilde{F}\left(\frac{t^{\beta}}{L^{\alpha}}\right)\right]^{(2\alpha-2)/\alpha}, \tag{9}$$

where the function $\tilde{F}$ has the same scaling as F.

Upon recalling that $\alpha=1/2$ and $\beta=1/3$, the above expression reduces to the one derived in Ref. [7]. In particular, we note that, since $F(z)$ tends to a constant for $z\to\infty$, the finite-size correction scale as $1/L$ whenever the analogy with KPZ dynamics holds. Conversely, in the infinite-space limit, the correction to the asymptotic value of the Lyapunov exponent decreases as $\delta\lambda(t)\approx t^{-2/3}$. Both predictions were successfully tested in Ref. [3]. Here we have derived the expressions for general values of α and β, and we shall see later on that they still hold in the case of Hamiltonian systems although with different values of the critical exponents. In particular, we will use relation (4) to find the scaling constant β, looking for the temporal growth of the interface width in a very large system. For a determination of the scaling constant α, we look at the spatial spectrum $S(k)$, and find γ from Eq. (7), thereby obtaining

$$\alpha=-\frac{\gamma+1}{2}. \tag{10}$$

III. MODELS

We consider a lattice of L oscillators with periodic boundary conditions described by the local displacement q_i and momentum p_i. The reference Hamiltonian reads

$$\mathcal{H}=\sum_i\left[\frac{p_i^2}{2}+V(q_{i+1}-q_i)+U(q_i)\right], \tag{11}$$

where $V(x)$ is the nearest-neighbor interaction potential, while $U(x)$ corresponds to the on-site potential. All variables are assumed to be scaled so as to be adimensional.

In the following we shall consider two main models: (i) the so-called Fermi-Pasta-Ulam β model, characterized by $V(x)=x^2/2-x^4/4$ and by the absence of external forces; and (ii) the Φ^4 model, characterized by harmonic interactions, $V(x)=x^2/2$, and by a double-well on-site potential, $U(x)=-x^2/2+x^4/4$.

The dynamical equations of the FPU model read

$$\dot{q}_i=p_i,$$
$$\dot{p}_i=(q_{i+1}-2q_i+q_{i-1})-(q_{i+1}-q_i)^3-(q_{i-1}-q_i)^3, \tag{12}$$

while the linearized equations for the evolution of an infinitesimal perturbation P_i and Q_i are

$$\dot{Q}_i=P_i,$$
$$\dot{P}_i=[1-3(q_{i+1}-q_i)^2](Q_{i+1}-Q_i) +[1-3(q_{i-1}-q_i)^2](Q_{i-1}-Q_i). \tag{13}$$

Similarly, the dynamical equations for the Φ^4 model are

$$\dot{q}_i=p_i,$$
$$\dot{p}_i=(q_{i+1}-2q_i+q_{i-1})+q_i-q_i^3, \tag{14}$$

and the corresponding equations for the perturbation read

$$\dot{Q}_i=P_i,$$
$$\dot{P}_i=Q_{i+1}-2Q_i+Q_{i-1}+(1-3q_i^2)Q_i. \tag{15}$$

The two models have been singled out as representative examples of two different classes of systems: the FPU equation describes the dynamics of an isolated system where the momentum is conserved as well as the energy; in the Φ^4 model, the presence of an on-site potential breaks the translational invariance and the conservation of the total momentum. Such a difference proves to be crucial, for instance, in determining heat-conduction properties, since heat conductivity is typically anomalous in the first case, while it is normal in the latter context [8].

In both cases we have decided to fix the same value of the energy density $E/L=5$. The two models have been simulated by implementing the so-called bilateral symplectic algorithm [9], with a time step $\Delta t=0.01$. This choice already proved good enough to guarantee that energy fluctuations remain smaller than 10^{-7}. Additionally, as a further check, some of the simulations were successfully compared with the McLachlan-Atela algorithm.

In analogy to Ref. [3], we introduce the local norm of the perturbation P_i, Q_i as $w_i(t)=\sqrt{P_i^2+Q_i^2}$, and call the field $w_i(t)$ the ''Lyapunov vector.'' We interpret its logarithm $H_i(t)=\ln w_i(t)$ as the height of a pseudointerface. In Fig. 1 we show a snapshot of the ''interface'' profile for the Φ^4 model. The profile looks definitely rough, although at first it seems to demonstrate larger deviations than one expects for a Brownian motion (in fact, some regions resemble Levy flights). Such large fluctuations, which correspond approximately to 27 orders of magnitude in the snapshot of Fig. 1, suggest the possibility of numerical problems in the integration of the profile. In practice, the accuracy of the integration depends on the relative amplitude of the perturbation in neighboring sites [see the structure of Eqs. (13) and (15)]. The approximate continuity of the Lyapunov vector makes such concerns almost immaterial. A more serious obstacle in the simulation of long chains is the possibility that either $|Q_i|$ or $|P_i|$ becomes so small that its squared value (needed to determine w_i) is smaller than the smallest useable number on the computer. In some of our simulations we have overcome this problem by either writing a suitable routine or integrating the equations in quadruple precision. For yet larger values of L, $|Q_i|$ and $|P_i|$ themselves can become too small, but we checked that this has never occurred in our simulations.

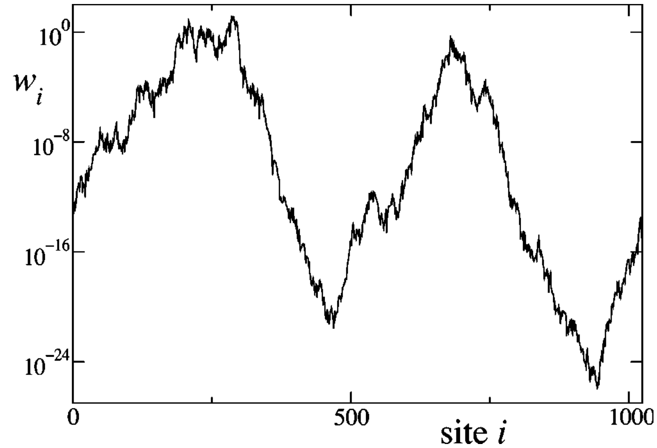

FIG. 1. A snapshot of the profile w_i for the Φ^4 model in a chain of length $L=1024$ after a long transient. The energy density is $E/L=5$, as in all other numerical studies presented in this paper.

IV. NUMERICAL EVIDENCE OF ANOMALOUS BEHAVIOR

A. Estimating the spatial exponent α

In order to estimate the spatial exponent α, we have made use of Eq. (10), determining γ from the power spectrum. The linear perturbation field has been allowed to evolve for various lattice lengths, until a stationary regime is attained for the witdh of the Lyapunov vectors. The spectra $S(k)$ for lengths $L=256$, 512, 1024, and 2048 in the FPU model are reported in Fig. 2. The very good overlap indicates that the lattice-size dependence is already quite negligible in this size range. In the inset, we report the logarithmic derivative $\gamma=\Delta(\log S)/\Delta(\log k)$, computed by looking at doubling values of the wave number. It converges (for $k\to 0$) to a value clearly below -2. As the exponent γ is connected to α by relation (10), we can conclude that the scaling behavior is definitely different from that expected for the KPZ universality class ($\gamma=-2$). However, a precise estimate of γ is a difficult task both because of statistical fluctuations that are very prominent in the small wave number regime, and in view of the observed residual dependence of the logarithmic

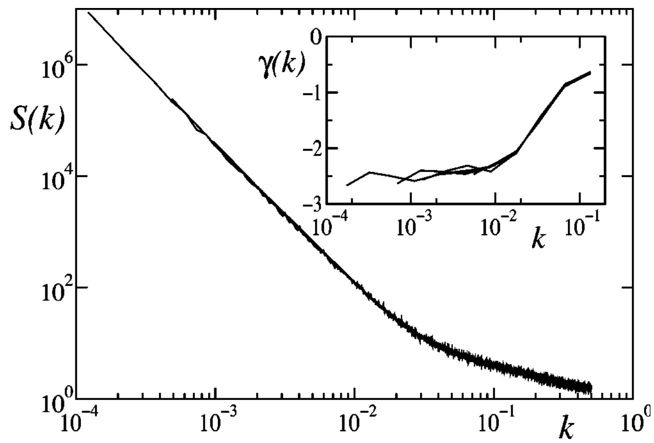

FIG. 2. The spatial power spectrum of the logarithmic profile H_i for the FPU model (after a sufficiently long transient to guarantee reaching the asymptotic state). The various curves refer to $L=256$, 512, 1024, and 2048. In the inset, we report the logarithmic derivative.

slope on k. A best fit of the almost linear region in Fig. 2 yields $\gamma=-2.5$. By approximately accounting for the systematic growth, we suggest that $\gamma=-2.6\pm0.1$, although the estimate of the error is quite uncertain, precisely in view of the slow but systematic dependence of the slope on k. The above value corresponds to $\alpha=0.8\pm0.05$, which can be interpreted as evidence of a superdiffusive (fractional Brownian) spatial profile.

The same scenario is qualitatively confirmed by the analysis of the data for the Φ^4 model (see Fig. 3). We again find a good overlap among the spectra for $L=2048$, 4096, and 8192, confirming that the system size in itself is not too small. On the other hand, the analysis of the logarithmic slope reveals a consistent dependence of γ on k. While we can definitely confirm that $\gamma<-2$ also in this case, an estimate of γ iteself is more problematic. We can conjecture that $\gamma=-2.8\pm0.2$. One should also note that $\gamma=-3$ is a limit value that corresponds to an exponential localization, where

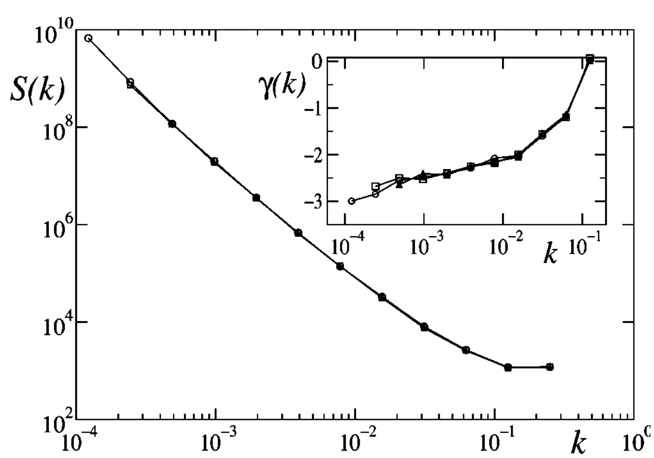

FIG. 3. The spatial power spectrum of the logarithmic profile H_i for the Φ^4 model in a statistically stationary state. The various curves refer to $L=2048$ (triangles), 4096 (squares), and 8192 (circles). The logarithmic derivative is shown in the inset.

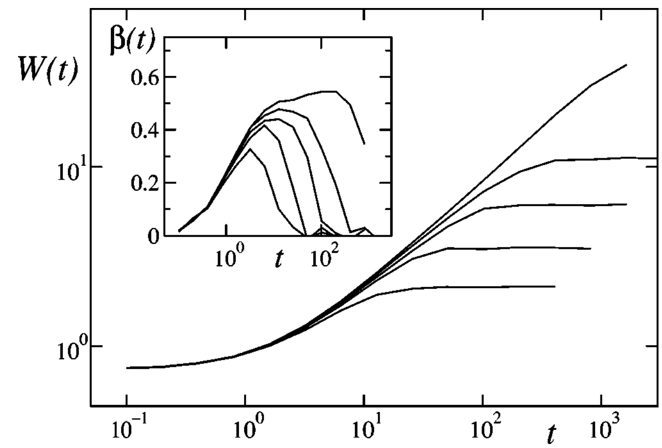

FIG. 4. Temporal growth of the width of the interface W for the chain lengths $L=256$, 512, 1024, 2048, and 8192 in the FPU model. Only for the largest lattice length is there an indication of a plateau for the logarithmic derivative (inset), in between the initial nonuniversal growth and saturation.

H_i consists of straight lines (this is the localization that one expects in disordered systems in the absence of any temporal dependence of the field variable).

B. Estimating the temporal exponent β

In order to find the temporal exponent β, we followed the development of the interface starting from a flat one [what corresponds to a spatially uniform initial linear perturbation $w_i(0)$]. Equation (4) can be tested by looking at the temporal evolution of the width $W(L,t)$. In Fig. 4 we have reported its dynamics for $L=256$, 512, 1024, 2048, and 8192 in the FPU model, while Fig. 5 refers to the Φ^4 problem [10]. One can see that for short times there is a slow growth that is essentially determined by the structure of the specific model. In addition, there is an intermediate range that grows with L, characterized by an almost power-law increase. Finally, we observe the saturation leading to a statistically stationary (in time) interface (the spectra in Figs. 2 and 3 were computed in this regime). The insets, wherein the logarithmic derivative

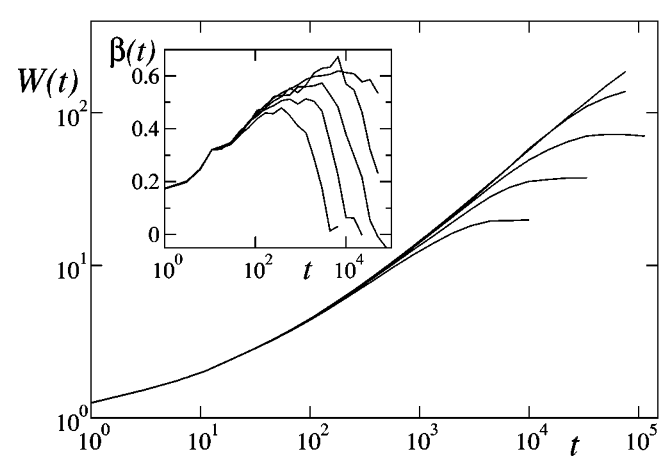

FIG. 5. Same as Fig. 4, but for the Φ^4 model. The chain lengths are 512–8192.

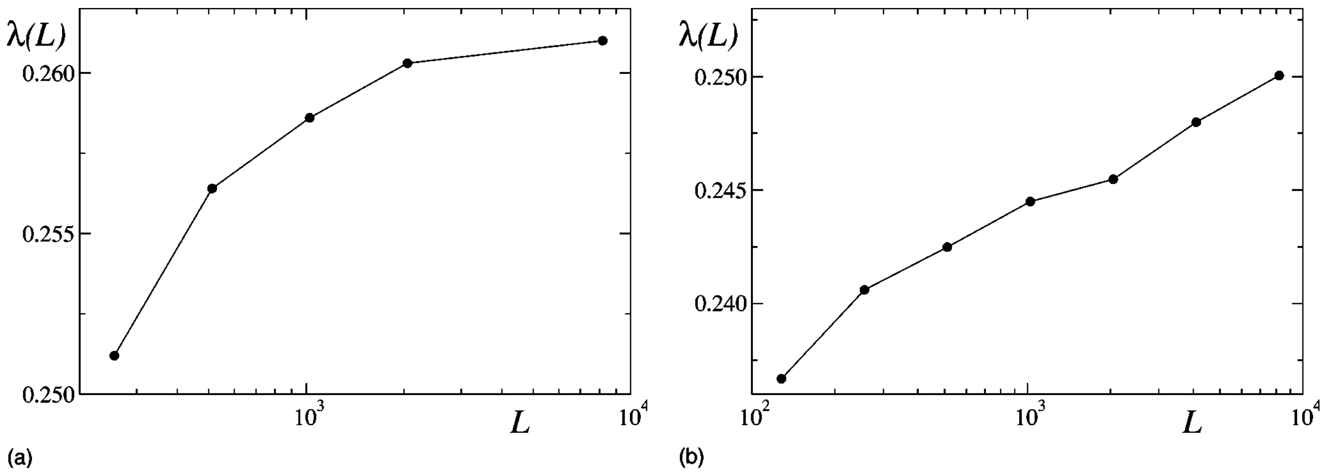

FIG. 6. The Lyapunov exponent $\lambda(L,\infty)$ vs the chain length for the FPU (a) and Φ^4 (b) models.

β is reported, show that yet larger lattice sizes would be needed in order to observe a clear plateau. In fact, the maximum value of the slope keeps increasing with L, and only for $L=8192$ can we find evidence of a constant-slope region. On the basis of these last results, we can estimate $\beta=0.55\pm0.05$ for the FPU model and $\beta=0.6\pm0.05$ for the Φ^4 model. Note, that the error is mainly statistical: there is no simple way to estimate the systematic deviations which are certainly visible if we look at the height of the curves in the insets. In any case, the estimated β values are definitely larger than 1/3, the value predicted by the theory for KPZ interfaces.

Before going on, let us note that although the spatial and temporal exponents are significantly different from the values expected for the KPZ equation, the same is not true for the dynamical exponent $z=\beta/\alpha$ that is remarkably close to 3/2 in both cases. By taking the central values for α and β, we obtain $z=1.45$ and 1.5 for the FPU and Φ^4 models, respectively. This is an indication that a single physical effect is responsible for the above observed anomalies.

C. Finite-size corrections to the Lyapunov exponent

We now conclude the discussion on the scaling properties of the Lyapunov vectors with the finite-size analysis of the Lyapunov exponent in the FPU model. The dependence of $\lambda(L,\infty)$ on L can be observed in Fig. 6(a). With the help of simulations performed with a longer chain ($L=8192$) and assuming a power-law convergence to the asymptotic value, we find that $\lambda_{as}:=\lambda(\infty,\infty)=0.265\pm0.001$, while the convergence rate is -0.5 ± 0.1. Before comparing with the theoretical expectations, let us note that the finite-size corrections are definitely larger in Hamiltonian problems compared to dissipative ones, and this should be taken as a serious obstacle to giving precise estimates. In view of this difficulty, we suspect that most of the estimates of the largest exponent obtained in one-dimensional systems should be carefully reexamined.

The results of a more detailed analysis of the finite-size deviations are reported in Fig. 7, where they are depicted versus a ratio of suitable powers of t and L in order to test the validity of Eq. (9). A reasonably good overlap has been found by reporting $t/L^{1.4}$ on the horizontal axis, and multiplying $|\lambda-\lambda_{as}|$ by $L^{0.45}$ (the large deviations observed for short times are absolutely irrelevant, as they concern the initial regime when no scaling behavior can be expected at all). The exponent 1.4 should be compared with the dynamical exponent z that, on the basis of previous simulations, turns out to be around 1.45. Such a difference is certainly compatible with the relatively large errors on the estimates of both α and β. On the other hand, the vertical scaling factor 0.45 should be compared with $2-2\alpha\approx0.4$: the deviation is the same as in the previous case, confirming that the inaccuracy is around 0.05. Once more, however, we want to warn against the existence of corrections that might coherently drift the various exponents towards slightly different values. The opportunity of considering larger lattices is specifically desirable for the Φ^4 model, since we could not perform a reliable finite-size scaling analysis: in that case, we limit ourselves to reporting in Fig. 6(b), the clearly slow dependence of the Lyapunov exponent on the system size.

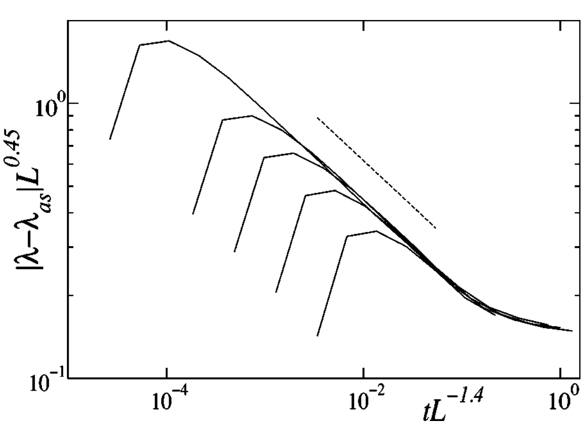

FIG. 7. Scaling behavior of the finite-size corrections of the Lyapunov exponent in the FPU model. The curves correspond to the systems of lengths $L=256$, 512, 1024, 2048, and 8192 (from bottom to top). The dashed line has a slope $-1/3$.

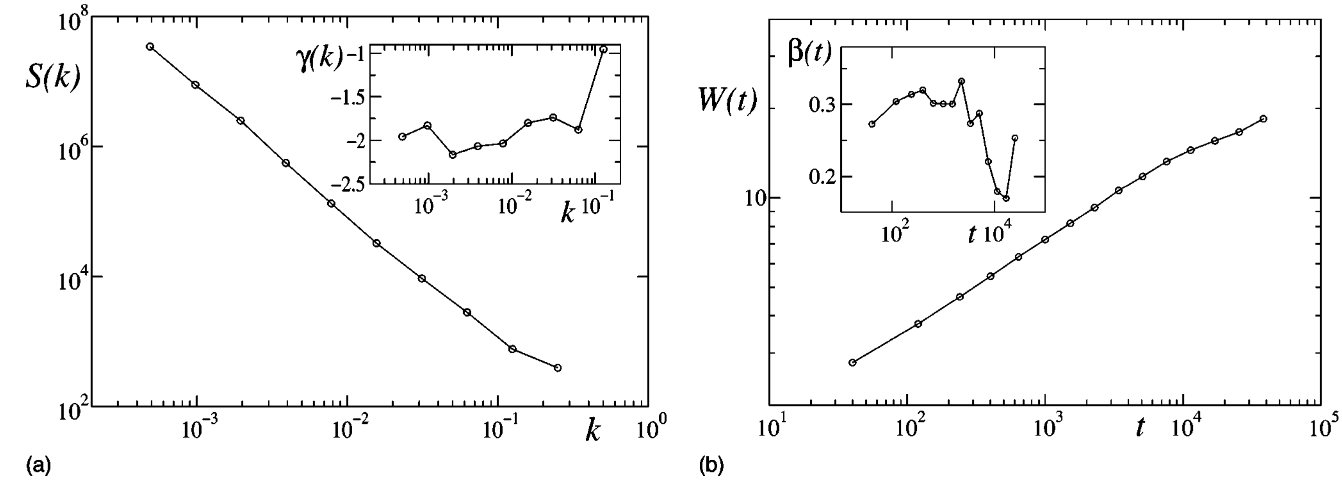

FIG. 8. Scaling properties of the field in the linearized Φ^4 model [Eq. (15)], when the multipliers $M_i(t)$ are reshuffled at regular intervals of time $T=40$ (a smooth connection between old and new multipliers was performed during the time interval $T_s=2$). The graphs should be compared with Figs. 3 and 5.

V. ORIGIN OF ANOMALOUS SCALING IN THE LYAPUNOV VECTOR

Having observed significant differences with respect to the KPZ-class behavior, we shall now discuss the possible sources for the observed anomalies. There are at least two reasons for these.

(i) The equation describing the dynamics of the Lyapunov vector [Eqs. (13) and (15)] is of wave type (second order in time), and thus qualitatively different from the KPZ equation, where inertia does not enter. It is therefore not clear whether $H_i=\ln\|w_i\|$ indeed satisfies an effective KPZ equation on sufficiently large spatial scales and over sufficiently long times.

(ii) The effective noise may have long-range correlations in space and/or in time. Indeed, the recent study of heat conductivity in one-dimensional chains has revealed that at least the correlations of the global heat flux decay with a power law in the FPU model [11]. On the other hand, this is not true in the Φ^4 model, where thermal conductivity is normal, so that this hypothesis needs a careful examination.

A. Tangent space dynamics in the presence of short range correlations

In order to test whether the discrepancies discussed in Sec. IV are due to the lack of a connection with the KPZ equation, we have decided to remove the possible long-range correlations potentially contained in the spatio temporal behavior of the multipliers

$$M_i(t) := 1-3(q_{i+1}-q_i)^2 \tag{16}$$

(for the FPU model) and

$$M_i(t) := 1-3q_i^2 \tag{17}$$

(for the Φ^4 case). In order to preserve as much as possible of the original structure, we have preferred to suitably modify the $M_i(t)$ in Eqs. (13) and (15) rather than replacing them with a random noise. More precisely, we have proceeded by first removing spatial correlations with a random reshuffling of the multipliers, i.e. by replacing $M_i(t)$ with $M_{j(i)}$, where $j(i)$ is a bijective random function from the interval $1,\dots,L$ onto itself. In order to get rid of temporal correlations, we further reshuffle [randomly change the function $j(i)$] every T time units. Finally, since the instantaneous switch from one to another given function introduces an undesired discontinuity, we have smoothed the connection between consecutive time intervals. As a result, all possible correlations are destroyed, and we are in the position to test whether the tangent-space dynamics does really imply a KPZ-class behavior. From Fig. 8, one can see that the expected scaling behavior is fully restored for what concerns both spatial and temporal scalings. Furthermore, we have calculated the skewness $\langle\overline{H}^3\rangle/W^3$ and the kurtosis $\langle\overline{H}^4\rangle/W^4-3$ of the interface profiles (where $\overline{H}=H-\langle H\rangle$). These quantities measure the closeness to the Gaussian distribution. In the original dynamics of the Lyapunov vector these quantities are already quite small, ≈ -0.06 and ≈ -0.4, respectively. After removing correlations, they are further reduced to ≈ 0.025 and ≈ -0.1, meaning that the distribution is Gaussian with a good accuracy. We can thus conclude that the first concern is unmotivated, and we should really attribute the anomaly to the existence of long correlations in the dynamics.

B. Role of correlations

Recent studies of the scaling behavior of the KPZ equation in the presence of either long spatial or temporal correlations showed that sufficiently long-ranged correlations can indeed modify the critical exponents (see Ref. [7], and references therein). As a particular example, we discuss the effect of long-range spatial correlations. Upon assuming that the spatial power spectrum $S_\xi(\kappa)$ of the noise term ξ in Eq. (2) behaves as

$$S_\xi(\kappa)=\kappa^{-2\psi} \tag{18}$$

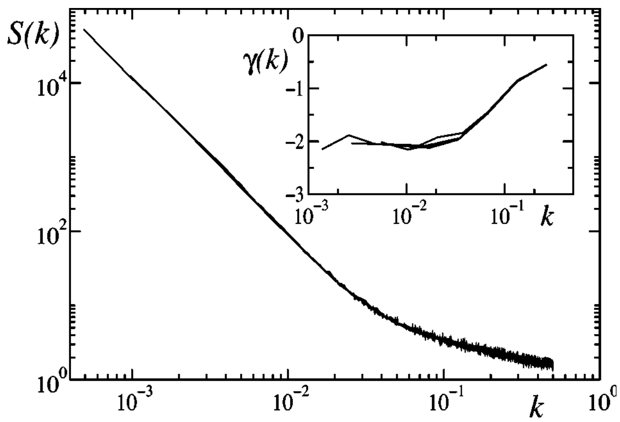

FIG. 9. Power spectrum of the logarithmic profile for the FPU model after a reshuffling of the multipliers. The logarithmic derivative is reported in the inset.

in the region of small wave numbers, it was found through renormalization group calculations [12] that, for $\psi > 1/4$,

$$\alpha = 1 - \frac{2}{3}(1-\psi), \quad \beta = \frac{1+2\psi}{5-2\psi}. \tag{19}$$

A blind use of the above formulas for the values of α and β presented in Sec. IV gives inconsistent results: from the first relation it follows $\psi \approx 0.7$, but the resulting value $\beta = 0.66$ is too large compared with our numerical findings. The picture does not change even if we refer to the theoretical predictions of the replica method [13], since this latter approach is equivalent to the former one for $\psi > 1/2$.

Since there is no direct way to connect the original tangent-space dynamics [Eqs. (13) and (15)] with an effective KPZ [Eq. (2)], we cannot directly estimate the correlation properties of the effective noise $\xi(x,t)$. However, we can demonstrate that the anomalous scaling cannot be explained with spatial correlations like Eq. (18) only. To get rid of spatial correlations we performed one single reshuffling of the multipliers M_i. In this procedure the temporal one-site correlations remain unaffected, while the spatial correlations are destroyed. The results for the power spectra in the FPU and Φ^4 problems are reported in Figs. 9 and 10, respectively. We see that the scaling exponents α and β are now closer to the KPZ values, but still there is a distinct difference. In particular, for γ we obtain values smaller than -2, what corresponds to $\alpha > 1/2$. We can, therefore, conclude that both spatial and temporal correlations of the multipliers $M_i(t)$ play crucial role in the anomalous scaling of the Lyapunov vector interface.

VI. DIRECT STUDY OF SPACE-TIME CORRELATIONS

In this section we report the results of direct studies of the correlation properties in the Hamiltonian lattices [Eqs. (12) and (14)]. Usually one looks at the simplest observables, namely, the variables $p_i(t)$, and $q_i(t)$ themselves. In our case, these observables do not exhibit any interesting behavior: in the Φ^4 model, no long-range correlations are present at all, while in the FPU model, the q_i's show a trivial diffusion due to the lack of external forces. Contrary to the properties of positions and momenta, in Ref. [11] it has been shown that in the FPU lattice the local heat flux is characterized by nontrivial correlations. This dependence on the observable indicates that the problem of characterizing the correlations is rather difficult [14]. In this paper, without pretending to derive general conclusions, we limit ourselves to studying the multipliers $M_i(t)$, defined as in Eqs. (16) and (17). All the correlation functions reported in this paper have been computed by Fourier-transforming a sufficiently long signal (pattern) sampled every 0.1 time units, and by afterwards averaging over at least 1000 different realizations.

Let us start by discussing the FPU case. In Fig. 11 we report a gray scale plot of the logarithm of the amplitude of the correlation function $C_M(i,t) = |\langle M_j(\tau) M_{j+i}(\tau+t)\rangle|$ of the multiplier. There one can see that the strongest correlations are either found along the line $i = vt$ [a best fit yields $v = 2.1(1)$], or for $i = 0$. This is consistent with the previous analysis of the local heat flux (see Ref. [11]) in the same model, which also revealed the existence of a propagation of correlations with a constant velocity that was traced back to

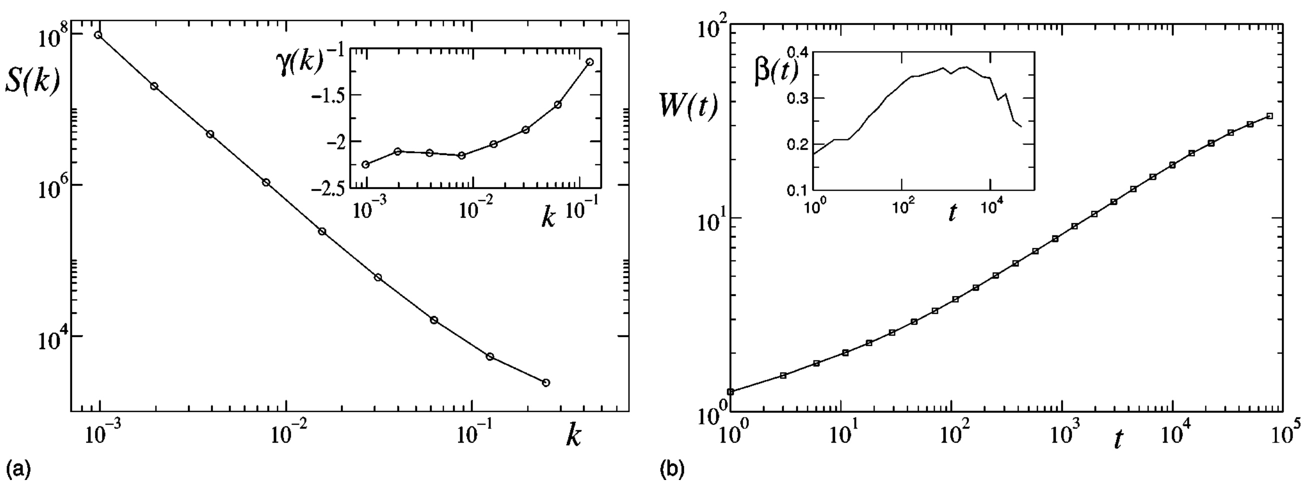

FIG. 10. The same as Fig. 8, but the reshuffling is performed only once; as a result, only spatial correlations are destroyed, but not temporal ones. The logarithmic derivatives (insets) still deviate from the standard values of the KPZ-class behavior.

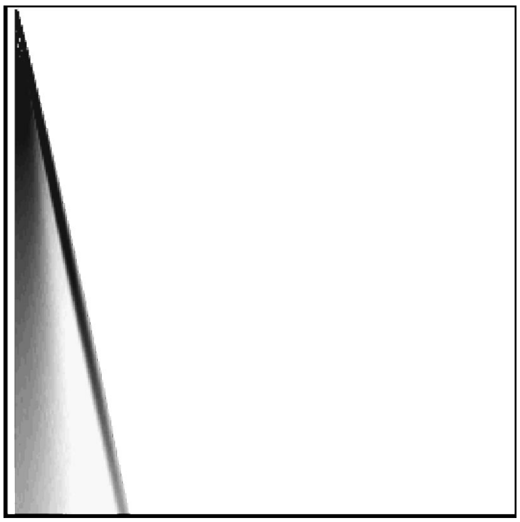

FIG. 11. Gray scale plot of the logarithm of the absolute value of the correlation function of the multiplier for the FPU model in a chain of 256 oscillators for a time span equal to 26.4. Space runs horizontally, while time flows vertically from top to bottom. Dark regions correspond to stronger correlations.

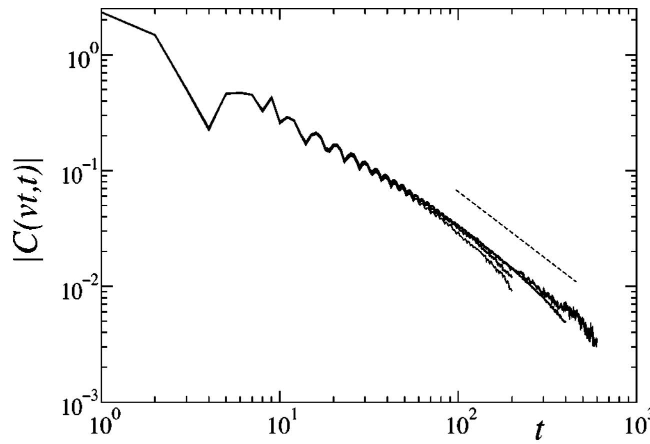

FIG. 13. Decay rate of temporal correlations of the multiplier in a moving frame with velocity $v=2.1(1)$ for chain lengths $L=128$, 256, 512, and 1024 in the FPU model (from bottom to top). The slope of the dashed line (drawn as a reference) is -1.2.

the quasi uncoupled behavior of the low wave number Fourier modes.

Therefore, there is no direct connection with the previous studies of KPZ dynamics in the presence of correlated noise, since the anomaly is neither purely spatial nor purely temporal. Looking more carefully at the pattern in Fig. 11, we find that equal-time spatial correlations (i.e., the uppermost horizontal slice in the figure) are δ like (the same also holds for the Φ^4 model). On the other hand, the standard temporal correlation in a given site decays as power law (see Fig. 12). Even though the rate cannot be accurately determined, one can confirm that such temporal correlations alone are not sufficient to explain the anomalous behavior of the Lyapunov vector, since the decay is nevertheless too fast to be compatible with the scaling behavior of the Lyapunov vector. Thus it is even more clear that correlations due to the propagation are crucial as well. A direct estimate of correlations along a properly moving frame yields the results reported in Fig. 13, where an apparent power-law decay is observed. Moreover, we observe that, as in Fig. 12, the decay is affected by strong finite-size corrections. This may be the reason for the difficulties encountered in our attempts to obtain precise estimates of the critical exponents from an interfacelike analysis of the Lyapunov vectors.

In the Φ^4 model we do not observe propagation of correlations with a definite velocity. In this case, there is only a slow decay of the one-site correlations $C_M(0,t)$. We illustrate this by reporting the temporal power spectrum $\Sigma(\omega)$ of the multiplier $M_i(t)$ in Fig. 14, where one can observe a low-frequency divergence, $\Sigma(\omega)\sim\omega^{\delta}$ with $\delta=-0.47\pm0.03$. From the approximate formula $\alpha=(0.22–0.845\delta)$, derived in Ref. [12] for purely temporal correlations, we find $\alpha\approx0.62$. Upon inverting Eq. (10), we find $\gamma\approx-2.2$. This result is in close agreement with the direct estimate of α after having removed spatial correlations (see the inset of Fig. 10).

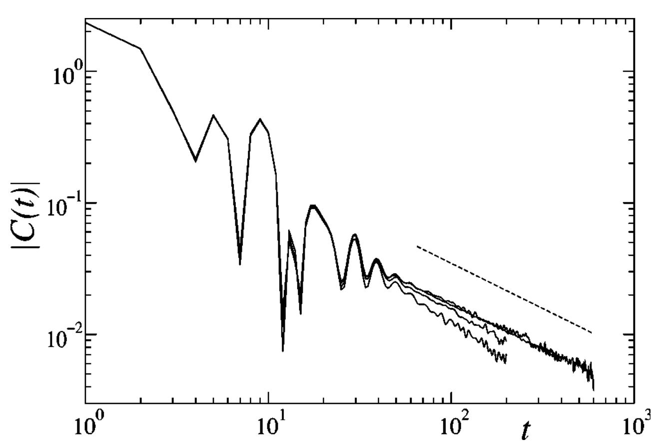

FIG. 12. One-site temporal correlations $C_M(0,t)$ of the multipliers $M_i(t)$ in the FPU model. Data for different chain lengths (from bottom to top, $L=128$, 256, 512, and 1024) demonstrate convergence to a power law decay with the exponent ≈-0.9 (dashed line).

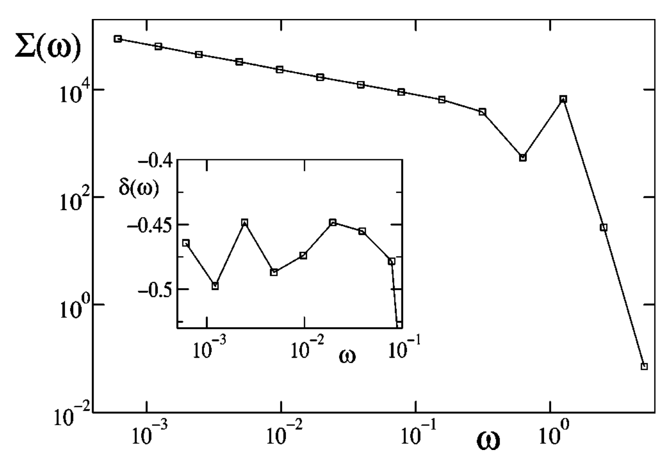

FIG. 14. Temporal power spectrum of the multipliers in the Φ^4 model for a lattice of length $L=2048$. The inset shows the local logarithmic derivative at small frequencies.

Therefore, we again confirm that also in the Φ^4 temporal correlations alone are not sufficient to account for the anomalous scaling exhibited by the Lyapunov vectors.

VII. CONCLUSIONS

In this paper we have carried on a detailed study of the scaling properties of the Lyapunov vector in two Hamiltonian (Fermi-Past-Ulam and Φ^4 models) chains, finding clear evidence of an anomalous behavior with respect to the previously studied one-dimensional space-time chaotic systems. Although the data presented are restricted to one set of parameters (including the density of energy), further selected simulations suggest that this anomaly exists at least in a broad parameter region if not everywhere. This anomaly is analogous to the divergence of the heat conductivity (in a broad subclass of one-dimensional Hamiltonian systems) in the sense that both phenomena can be traced back to the existence of long-range correlations of suitable observables. A deeper understanding of such anomalies can therefore arise only from a better comprehension of spatiotemporal correlations. In Ref. [11], it was shown that in the FPU model they are connected with the ''hydrodynamic'' behavior of the long-wavelength Fourier modes, and that self-consistent mode-coupling theory enables one to predict the correct divergence rate of the heat conductivity. Such an approach is, however, insufficient to describe the whole space-time structure of the multiplier correlations. In particular, it does not predict any slow decay of correlations in the Φ^4 model, but also fails to predict the correct decay of the purely temporal correlations of multipliers in the FPU case. So far, one can only suspect that the existence of conservation laws is responsible for long-range correlations: energy is conserved in both the FPU and Φ^4 models, and its diffusion properties might be the ultimate source for the similar behaviors exhibited by two otherwise rather different models. However, only when a global coherent description of the long-time properties of the relevant observables will be obtained, it will be possible to establish whether the near equality of the scaling exponents observed in FPU and Φ^4 models is indeed an indication that both belong to the same universality class.

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